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APPLICATION OF TCHÉBYCHEF'S PRINCIPLE IN THE PROJECTION OF MAPS*

BY G. W. HILL

THE question how to obtain the best map of a portion of the surface of a sphere or ellipsoid of revolution is certainly an important one. Tchébychef was, I believe, the first mathematician to consider this matter from a general point of view. His dictum is "A map is the best when, within its limits, the logarithm of the scale has the least possible oscillation." It is here assumed that this function of the scale is a measure of its own importance. It may be objected that the function enjoying this property is not the logarithm. This, however, has no more validity than the objections made against the method of least squares in the treatment of discordant observations. The author evidently was induced to adopt his principle by the elegant transformation it is capable of.

In the projection of the oblate ellipsoid of revolution on a plane, let x and y denote the rectangular coordinates of the projected point, u the longitude and z the geographical colatitude of the point to be projected. Let ϵ denote the eccentricity of the meridian of the ellipsoid. To avoid the consideration of a linear unit we assume that the ellipsoid has the semi-axis 1, and that the scale on the outer contour of the map is also 1. The multiplications to be made for other conditions are so obvious that they need no mention. As we are not restricted to the use of z and u for defining the position of a point on the ellipsoid, and as the relative properties of this surface and the plane, in respect to projection, demand another variable in the place of z, let us put

$$v = \log\left[\left(\frac{1 + \epsilon\cos z}{1 - \epsilon\cos z}\right)^{\frac{\epsilon}{2}} \tan\frac{z}{2}\right],$$

where the logarithm is hyperbolic. While z goes from 0 to π, v goes from $-\infty$ to $+\infty$. Then, according to the theory of *conforme* projection, the connection between the variables x, y and u, v is shown by the equation

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* See the preceding paper.

$$x + y\sqrt{-1} = f(u + v\sqrt{-1}),$$

f denoting any arbitrary function. The scale m is given by the equation

$$\log m = \log \frac{\sqrt{1 - \epsilon^2 \cos^2 z}}{\sin z} + \tfrac{1}{2} \log f'(u + v\sqrt{-1}) + \tfrac{1}{2} \log f'(u - v\sqrt{-1}),$$

where f' denotes the derivative function of f.* For brevity let log m be denoted by V and its first term by A, and the two following terms involving the arbitrary function by W, so that

$$V = A + W.$$

In establishing his principle Tchébychef compares it to the statical case of the convection of heat. But we may, with advantage, substitute for the latter the general principle of hydrostatics.

Let V_1 denote the greatest value assumed by V in the portion of the surface to be projected, and V_0 the least. Then, in order that the map may be the best possible, $V_1 - V_0$ must be a minimum. This is expressed in the language of the calculus of variations by $\delta(V_1 - V_0) = 0$, the variation having reference to the form of the function f. (The criterion for discriminating a minimum from a maximum need not be noticed).

Now, to form the cognate problem in hydrostatics, let us suppose that, in the plane of u, v, the surface to be projected is covered with a fluid of infinitesimal depth, such that the gravitational action of itself on itself has a potential V. The fluid can be in equilibrium if V is constant along the boundary of the surface. In case the boundary consists of several detached portions, the theorem still holds, but the constants belonging to each portion may be different. Appealing to the principle of least action for the condition of equilibrium, if V_1 is the maximum of the potential and V_0 the minimum, just as before, $\delta(V_1 - V_0) = 0$, the δ having reference to the distribution of the density ρ of the fluid. Now Poisson's equation is

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} + 4\pi\rho = 0.$$

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* For demonstration of these equations (see the preceding paper) consult Gauss, *Werke*, vol. 4, p. 193. The essential parts of Gauss' investigation have been translated by Thomas Craig in his *Treatise on Projections* (separate publication of the U. S. Coast and Geodetical Survey, Washington, 1882). At the end of the supplementary volume of Boole's *Differential Equations* will be found a brief treatment of the matter.

But, in the first problem, we have

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = \frac{\partial^2 A}{\partial u^2} + \frac{\partial^2 A}{\partial v^2}.$$

Hence, if we make

$$4\pi\rho = -\left(\frac{\partial^2 A}{\partial u^2} + \frac{\partial^2 A}{\partial v^2}\right),$$

the two problems are rendered identical.

Thus the transformation of Tchébychef's dictum is "That map is best in which the scale is constant along its boundary; it being understood that when the latter consists of disconnected portions the constant may vary from one portion to another."

The Tchébychef principle demands that, with our conventions, V should vanish on the contour of the map. It has become the fashion recently to allow meridians and parallels to form this contour. Consequently it is proposed to treat here the simple case where the portion of the ellipsoidal surface to be projected is contained between portions of two parallels and two meridians. By counting the longitudes from the middle meridian, we can suppose that $\mp u_0$ are the limiting values of u; also let v_0 and v_1 be the lower and upper limits of v; and z_0, z_1 the corresponding values of z. The problem, thus limited, has the advantage that the contour is regarded as dependent on u, v instead of x, y.

A is stated in terms of z instead of v; the first should be replaced by the second; but the equation connecting them is eminently transcendental, the infinite series in powers of ϵ^2 is complicated. Limiting ourselves to the first power,

$$\log\frac{\sqrt{1-\epsilon^2\cos^2 z}}{\sin z} = \log\cosh v + \epsilon^2 \tanh v(1 - \tfrac{1}{2}\tanh v).$$

It is readily seen that our problem is a case of what is known as the Dirichlet problem. In the mode of solving this, started by Fourier, it is necessary to suppose that

$$\log\frac{\sqrt{1-\epsilon^2\cos^2 z}}{\sin z} = A + Bv + \sum_{i=1}^{i=\infty} N_i \sin i\,\frac{\pi(v-v_0)}{v_1-v_0},$$

A, B, N_i being constants. A and B are evidently determined by the equations

$$A + Bv_0 = \log \frac{\sqrt{1 - \epsilon^2 \cos^2 z_0}}{\sin z_0}, \qquad A + Bv_1 = \log \frac{\sqrt{1 - \epsilon^2 \cos^2 z_1}}{\sin z_1}.$$

The expression

$$W = \tfrac{1}{2} \log f'(u + v\sqrt{-1}) + \tfrac{1}{2} \log f'(u - v\sqrt{-1})$$

satisfies the differential equation

$$\frac{\partial^2 W}{\partial u^2} + \frac{\partial^2 W}{\partial v^2} = 0.$$

A particular integral of this is $W = UV$, where U is a function of u alone, and V of v alone. By the substitution of this the differential equation becomes

$$V \frac{\partial^2 U}{\partial u^2} + U \frac{\partial^2 V}{\partial v^2} = 0.$$

This may be satisfied in two different ways, shown by the equations

$$\frac{\partial^2 U}{\partial u^2} = \pm\, cU, \qquad \frac{\partial^2 V}{\partial v^2} = \mp\, cV,$$

where c is a constant. Thus both U and V are periodic functions, one to a real, the other to an imaginary period. Which is to have the real, and which the imaginary period, depends on the circumstances of the problem. Here U is to have the imaginary and V the real period. Thus the particular integral may be written

$$W = M \cosh(cu + k) \sin(cv + h),$$

where M, k, h are the arbitrary constants. As the equation is linear, a more general integral may be written as

$$W = \sum \left[M \cosh(cu + k) \sin(cv + h) \right],$$

where the summation extends to all permissible values for c. But, in this case, W must have the same value when the sign of u is reversed, thus $k = 0$; also W must vanish when $v = v_0$ or v_1; this demands that c must be an integral multiple of $\pi/(v_1 - v_0)$. Also it is lawful to add $- A - Bv$ to the expression for W. All conditions will be fulfilled if we write

$$W = -A - Bv + \sum_{i=1}^{i=\infty} M_i \cosh(icu)\sin[ic(v - v_0)].$$

Here we have $c = \pi/(v_1 - v_0)$, and A and B have the constant values previously attributed to them.

Thus the proper form for $\log m$ is

$$\log m = \sum_{i=1}^{i=\infty} \left\{ N_i \sin ic(v - v_0) + M_i \cosh(icu) \sin ic(v - v_0) \right\}.$$

Thus $\log m$ clearly reduces to 0 when $v = v_0$ or v_1; and it will reduce to 0 when $u = u_0$, provided that M_i is determined by the equation

$$M_i \cosh(icu_0) = -N_i.$$

Hence

$$\log m = \sum_{i=1}^{i=\infty} \left[1 - \frac{\cosh(icu)}{\cosh(icu_0)} \right] N_i \sin[ic(v - v_0)].$$

By equating the two expressions for W we have

$$\tfrac{1}{2}\log f'(u + v\sqrt{-1}) + \tfrac{1}{2}\log f'(u - v\sqrt{-1})$$
$$= -A - Bv + \sum_{i=1}^{i=\infty} M_i \cosh(icu) \sin[ic(v - v_0)].$$

In attempting to derive the form of the function f from this equation some difficulties have been met with, and a method, suggested by a graphical treatment, has been substituted. The exposition of this is better understood in the elaboration of an example.

For this let us assume $z_0 = 40°$, $z_1 = 50°$, $u_0 = 5°$. The elements of the Clark spheroid give $\log_{10}\epsilon = 8.916\ 3655\ 961$. As showing the connection between the variables z and v we employ the equation

$$v = \log\tan\frac{z}{2} + \epsilon^2\cos z + \tfrac{1}{3}\epsilon^4\cos^3 z + \tfrac{1}{5}\epsilon^6\cos^5 z + \cdots.$$

Thence it results that

$v_0 = -1.00547\ 07238$, $v_1 = -0.75853\ 60483$, $\log_{10} c = 1.10456\ 77942$,

$c = 12.72236\ 3332$, $c \times 1^0 = 0.22204\ 71288$.

If s be an argument which goes from 0 to π when v goes from v_0 to v_1, we have the following table of correspondence between v and z:

$12s$	v	z	$\log\frac{\sqrt{1-\varepsilon^2\cos^2 z}}{\sin z}$	$A+Bv$	Difference
0	$-$1.00547 07238	40°	0.43994 0702	0.43994 0702	0
π	98489 28342	40 45′57″.7853	42426 5941	42537 1276	$-$0.00110 5335
2π	96431 49446	41 32 39 .2681	40877 2264	41080 1851	202 9587
3π	94373 70549	42 20 4 .6247	39346 5399	39623 2425	276 7026
4π	92315 91653	43 8 14 .0019	37835 1192	38166 3000	331 1808
5π	90258 12757	43 57 7 .5135	36343 5594	36709 3574	365 7980
6π	88200 33861	44 46 45 .2414	34872 4649	35252 4149	379 9500
7π	86142 54964	45 37 7 .2306	33422 4519	33795 4723	373 0204
8π	84084 76068	46 28 13 .4891	31994 1444	32338 5297	344 3853
9π	82026 97172	47 20 3 .9873	30588 1761	30881 5872	293 4111
10π	79969 18276	48 12 38 .6544	29205 1884	29424 6446	219 4562
11π	77911 39379	49 5 57 .3775	27845 8308	27967 7021	$-$0.00121 8712
12π	75853 60483	50	26510 7595	26510 7595	0

The values of A and B are, $A = -0.2719\,4625$, $B = -0.7080\,1360$. The numbers in the last column of the table are to be represented by a periodic function of the argument s, and we find

$$\begin{aligned}
&-0.00392\,1901 \sin s &&+ 0.00008\,6185 \sin 2s \\
&-0.00014\,5846 \sin 3s &&+ 0.00001\,0693 \sin 4s \\
&-0.00003\,1124 \sin 5s &&+ 0.00000\,3086 \sin 6s \\
&-0.00001\,0791 \sin 7s &&+ 0.00000\,1187 \sin 8s \\
&-0.00000\,4296 \sin 9s &&+ 0.00000\,0443 \sin 10s \\
&-0.00000\,1185 \sin 11s.
\end{aligned}$$

The values of the common logarithms of the hyperbolic cosine of various multiples of $c \times 1^0$ are given in the following table:

i		i		i		i	
1	0.0106195	9	0.5747815	20	1.6277072	32	2.7848529
2	0.0414879	10	0.6683966	21	1.7241194	33	2.8812866
3	0.0899754	11	0.7630127	22	1.8205394	35	3.0741546
4	0.1526127	12	0.8582766	24	2.0133924	36	3.1705884
5	0.2258977	14	1.0499092	25	2.1098226	40	3.5563233
6	0.3068064	15	1.1460329	27	2.3026865	44	3.9420586
7	0.3929821	16	1.2422634	28	2.3991193	45	4.0384924
8	0.4827065	18	1.4349258	30	2.5919860	50	4.5206616
						55	5.0028307

The values of the latter factors of the terms of log m are easily derived from

$c(v - v_0)$	$c(v - v_0)$	$c(v - v_0)$	$c(v - v_0)$
0°	57° 27′ 5″.44	111° 44′ 32″.24	163° 20′ 3″.70
19 32′ 3″.44	75 52 30.43	129 13 8 .65	180
38 40 39 .54	93 57 58.82	146 24 42 .19	

The scale along the contour of the map being 1.000 0000, the deviations of it for each degree of colatitude and longitude within the area of the map in units of the seventh decimal are given in the following table:

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$	$u = 5°$
40°	0	0	0	0	0	0
41	− 9820	− 9676	− 9212	− 8307	− 6618	0
42	16882	16618	15775	14155	11175	0
43	21685	21334	20226	18112	14254	0
44	24485	24088	22824	20418	16054	0
45	25377	24956	23646	21152	16627	0
46	24433	24041	22789	20402	16084	0
47	21625	21285	20200	18114	14291	0
48	16758	16505	15692	14119	11191	0
49	− 9669	− 9534	− 9097	− 8240	− 6604	0
50	0	0	0	0	0	0

The corresponding table for all values of u, in the case of Species II of Lagrange's projection, when the exponent is adopted so that the scale is equal along the northern and southern limits of the map, stands thus:

z		z		z		z	
40°	0	43°	– 32283	46°	– 36233	49°	– 13395
41	– 14032	44	36659	47	31538	50	0
42	– 24766	45	– 37956	48	– 23914		

The sum of all the deviations in the case of the map on Tchébychef's principle is 0.0754149; but, in the latter table the corresponding sum is 0.1504656; thus nearly double the previous one. Hence it may be judged how much the first map is to be preferred.

The method of constructing the map on Tchébychef's principle, here employed, depends on drawing each meridian and parallel at intervals of 1°. In the first place it is necessary, from the preceding table to derive the mean value of the scale in passing over a degree of longitude or colatitude. The rule adopted for finding this is to interpolate into the middle, and, differencing the resulting table, add one-sixth the second difference. Between $u = 4°$ and $u = 5°$ the change of differences is very rapid, and one must find means of estimating the mean independent of this rule. We have:

VALUES OF THE MEAN SCALE ALONG DEGREES OF THE MERIDIANS

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$
40°	0.9994875	94949	95187	95649	96530
41	86426	86632	87293	88574	90942
42	80537	80847	81832	83717	87168
43	76741	77118	78313	80590	84731
44	74928	75335	76638	79103	83577
45	74944	75349	76643	79100	83549
46	76810	77179	78356	80609	84709
47	80634	80934	81891	83738	87146
48	86576	86772	87404	88633	90946
49	94994	95063	95284	95722	96563
50					

VALUES OF THE MEAN SCALE ALONG DEGREES OF THE PARALLELS

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$
41°	0.9990227	90524	91189	92504	95485
42	83206	83748	84946	87195	92354
43	78432	79147	80716	83598	90309
44	75647	76460	78263	81527	89120
45	74763	75622	77474	80885	88830
46	75698	76502	78290	81528	89277
47	78488	79184	80729	83603	90282
48	83326	83846	85019	87202	92407
49	90376	90754	91288	92535	95597

According to the dimensions of the Clark spheroid the curved distances along the meridians and parallels in miles are:

z	Meridians	Parallels	Reduced to Chords on the Map						
			0°	1°	2°	3°	4°	5°	
40°		$44^{m}.55229$							$44^{m}.55201$
	$69^{m}.10934$		$69^{m}.10934$	34	33	33	32	31	
41		45.46947							45.46919
	69.09711		69.09711	11	10	10	09	08	
42		46.37268							46.37239
	69.08483		69.08483	83	82	82	81	80	
43		47.26165							47.26136
	69.07251		69.07251	51	50	50	49	48	
44		48.13610							48.13580
	69.06017		69.06017	17	16	16	15	14	
45		48.99579							48.99549
	69.04784		69.04784	84	83	83	82	81	
46		49.84045							49.84014
	69.03551		69.03551	51	50	50	49	48	
47		50.66984							50.66953
	69.02320		69.02320	20	19	19	18	17	
48		51.48370							51.48338
	69.01093		69.01093	93	92	92	91	90	
49		52.28180							52.28148
	68.99873		68.99873	73	72	72	71	70	
50		53.06389							53.06356

The reductions from the arcs to the chords are very minute, at most only 33 units of the last decimal for the parallels and 3 units for the meridians.

Desiring to construct the map on a scale of 100^m to the inch, we have only to divide the numbers just given by 100 and multiply by the mean scale to get the distances between the projected points on the map. Thus is obtained the following table:

DISTANCES ALONG THE MERIDIANS

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$	$u = 5°$
40°	$0\overset{\text{in}}{.}6907394$	7444	7606	7927	8533	.0931
41	6900332	0475	0930	1814	3451	9708
42	6895036	5251	5932	7233	9616	.8480
43	6891184	1444	2270	3843	6703	.7248
44	6888703	8984	9882	.1584	.4673	.6014
45	6887484	7763	8654	.0352	.3424	.4781
46	6887541	7797	8606	.0163	.2992	.3548
47	6888954	9160	9819	.1094	.3444	.2317
48	6891829	1965	2398	3248	4841	.1090
49	6896419	6467	6617	6919	7500	9870
50						

DISTANCES ALONG THE PARALLELS

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$
40°	$0\overset{\text{in}}{.}4455201$	55201	55201	55201	55201
41	4542504	42639	42941	43540	44895
42	4629481	29731	30287	31330	33722
43	4715942	16280	17022	18383	21555
44	4801857	02249	03118	04648	08343
45	4887184	87605	88512	90184	94075
46	4971901	72302	73194	74807	78659
47	5056053	56406	57190	58645	62029
48	5139752	40021	40625	41748	44428
49	5223117	23314	23588	24246	28547
50	5306356	06356	06356	06356	06356

We have now the sides of 50 quadrilaterals, which must be so disposed that they continuously cover the plane of projection, and moreover, must fit against one of the projected parallels. We take for the latter the one which corresponds to $z = 50°$. By subtracting the distances along the parallel of $z = 49°$ from those of $z = 50°$ we obtain the quantities in the first column of the following table, and, by comparing them with the distances along the meridians, we obtain the inclinations of two adjacent meridians at the beginning of their courses. By subjecting these to integration we have the inclination of each meridian to the middle meridian. Multiplying the sine and cosine of this by 0.5306356 we have the increments of the coordinates x and y which we denote by Δx and Δy. Thus we have the following table:

u	Differences	Inclination	Integrated	Δx	Δy
1°	0.0083239	41′ 29″.46	0° 20′ 44″.73	+ 0.0032022	+ 0.5306260
2	83042	41 23 .52	1 2 11 .22	95984	5305488
3	82768	41 15 .21	1 43 30 .58	159749	5303951
4	82110	40 55 .32	2 24 35 .85	223127	5301663
5	80509	40 6 .62	3 5 6 .82	285595	5298665

Having now the coordinates of angles of the quadrilaterals which abutt against the parallel of $z = 50°$, all the quadrilaterals can be made to fill continuously the plane of projection in determinate fashion. The rigorous following out of this procedure requires the determination of one of the diagonals of each quadrilateral, whence the position of the two unknown sides is derived. But the little curvature in the meridians enables us to greatly shorten the labor. Each meridian may be regarded as the envelop of all the circles described from the known corners of the quadrilaterals just beneath it with the appropriate radii given in a preceding table. Thus it is easy to compute the inclination of the two sides of each quadrilateral which form parts of two consecutive meridians, and by addition to get the inclination on the middle meridian. Thence are determined the changes of the coordinates in passing along the meridian from one parallel to the following. The table shows these inclinations and the values of Δx and Δy:

	$u = 1^\circ$			$u = 2^\circ$		
z	Inclination	Δx	Δy	Inclination	Δx	Δy
50°						
	41′ 29″.46	+.6895963	−.0083233	1° 22′ 52″.98	+.6894613	−.0166259
49						
	41 34 .86	6891460	83359	23 7 .41	6890387	166640
48						
	41 45 .87	6888652	83693	23 28 .99	6887787	167298
47						
	41 59 .92	6887283	84146	23 58 .12	6886561	168241
46						
	42 16 .86	6887243	84710	24 35 .78	6886568	169499
45						
	42 34 .69	6888457	85321	25 9 .39	6887768	170699
44						
	42 51 .35	6890911	85908	25 44 .01	6890127	171868
43						
	43 6 .26	6894710	86454	26 15 .38	6893762	173007
42						
	43 19 .72	6899927	86970	26 42 .71	6898733	174047
41						
	43 26 .83	6906892	87296	26 57 .63	6905397	174715
40						

	$u = 3^\circ$			$u = 4^\circ$		
z	Inclination	Δx	Δy	Inclination	Δx	Δy
50°						
	2° 4′ 8″.19	+.6892424	−.0248993	2° 45′ 3″.51	+.6889551	−.0331046
49						
	4 29 .77	6888727	249581	45 37 .65	6886840	332059
48						
	5 6 .25	6886531	250721	46 32 .74	6885356	333830
47						
	5 52 .52	6885544	252231	47 41 .16	6884795	336092
46						
	6 50 .64	6885662	254179	49 2 .60	6885090	338832
45						
	7 48 .60	6886822	256159	50 27 .42	6886200	341725
44						
	8 39 .89	6889014	257956	51 39 .75	6888106	344241
43						
	9 29 .09	6892341	259727	52 51 .42	6890897	346781
42						
	10 12 .96	6896865	261366	53 55 .85	6894616	349127
41						
	10 37 .33	6902940	262413	54 35 .68	6899624	350717
40						

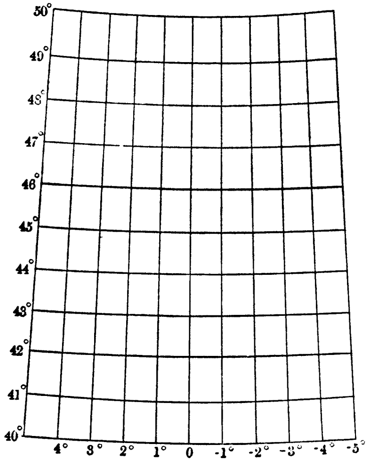
	$u = 5°$		
z	Inclination	Δx	Δy
50°			
	3° 25′ 10″.13	+ .6887585	−.0411548
49			
	26 11 .05	6888681	413655
48			
	27 34 .99	6889739	416533
47			
	29 11 .98	6890770	419847
46			
	31 9 .23	6891760	423840
45			
	33 7 .89	6892746	427881
44			
	34 51 .29	6893762	431423
43			
	36 33 .69	6894776	434913
42			
	38 7 .31	6895803	438120
41			
	39 12 .55	6896886	440379
40			

By the addition of the values of Δx and Δy we arrive at the following values of the rectangular coordinates of the intersections of the meridians and parallels:

	$u = 0°$		$u = 1°$		$u = 2°$	
Lat.	x	y	x	y	x	y
	in		in	in	in	in
40°	0.000000	0	0.003202	0.530626	0.012801	1.061175
41	0.689642	0	0.692798	522303	0.702262	1.044549
42	1.378825	0	1.381944	513967	1.391301	1.027885
43	2.067720	0	2.070810	505597	2.080079	1.011155
44	2.756474	0	2.759538	497183	2.768735	0.994331
45	3.445223	0	3.448262	488712	3.457392	0.977381
46	4.134093	0	4.137108	480180	4.146169	0.960314
47	4.823211	0	4.826199	471589	4.835182	0.943127
48	5.512715	0	5.515670	462944	5.524558	0.925827
49	6.202748	0	6.205663	454247	6.214431	0.908422
50	6.893488	0	6.896352	445517	6.904971	0.890950

	$u = 3°$		$u = 4°$		$u = 5°$	
Lat.	x	y	x	y	x	y
	in	in	in	in	in	in
40°	0.028775	1.591570	0.051088	2.121736	0.079648	2.651603
41	0.718018	1.566671	0.740043	2.088632	0.768406	2.610448
42	1.406891	1.541712	1.428727	2.055426	1.457274	2.569082
43	2.095544	1.516640	2.117263	2.022043	2.146248	2.527436
44	2.784098	1.491417	2.805742	1.988433	2.835325	2.485444
45	3.472664	1.465999	3.494251	1.954550	3.524501	2.443060
46	4.161346	1.440383	4.182871	1.920385	4.213776	2.400282
47	4.850248	1.414588	4.871682	1.885954	4.903152	2.357130
48	5.539482	1.388615	5.560772	1.851276	5.592630	2.313629
49	6.229168	1.362479	6.250233	1.816363	6.282210	2.269827
50	6.919462	1.336237	6.940196	1.781291	6.971898	2.225789

These numbers serve for plotting the appended map, which covering an area of about 700 miles by 500 miles, has a variation of scale of about a quarter of one per cent. The scale on the boundary is 300^m to the inch. Such a map is interesting as being the best possible if we admit Tchébychef's principle.



Reseau of Meridians and Parallels constructed on Tchébychef's Principle.